# *The Parametric Representation of the Tetrahedroid Surface.*

By Derrick N. Lehmer.

1. Weber* has discussed the 16-nodal Kummer Surface by means of the double theta functions. The tetrahedroid surface is a special case of the Kummer Surface noticed by Cayley,† and discussed by him by its representation in tetrahedral coordinates. Study‡ has expressed the surface parametrically by means of certain $\Theta$-functions defined by him and closely related to the $\vartheta$-functions of Jacobi. He derives also the Cartesian equation in a very elegant form.

Cayley has observed that the tetrahedroid is a homographic projection of Fresnel's Wave-Surface. The latter has been discussed by Lacour|| who represents the surface parametrically using the *sn*, *cn* and *dn* elliptic functions.

2. We may represent the tetrahedroid surface parametrically as follows :§

$$
\begin{aligned}
x_0 &= \sigma\, u\, \bar{\sigma}\, v, \\
x_1 &= \sigma_1 u\, \bar{\sigma}_1 v, \\
x_2 &= \sigma_2 u\, \bar{\sigma}_2 v, \\
x_3 &= \sigma_3 u\, \bar{\sigma}_3 v.
\end{aligned}
$$

$x_0$, $x_1$, $x_2$, $x_3$ are the four homogeneous coordinates of a point in space. $u$ and $v$ are independent parameters $\sigma_\lambda$ $\bar{\sigma}_\lambda$ are the $\sigma_\lambda$-functions of Weierstrass built on

* Weber, Crelle's Journal, Vol. 84, p. 349.

† Cayley, Collected Works, Vol. I, p. 302, Vol. V, p. 431 and Vol. X, p. 437.

‡ Study, "Spharische Trigonometrie, Orthogonal Substitutionen und Elliptische Functionen," p. 225.

|| Lacour, Nouvelles Annales, XVII (3), p. 266.

§ See Hutchinson's Thesis On the Reduction of Hyperelliptic Functions, p. 36. Also Bricard, Nouvelles Annales, 3d series, Vol. 18, p. 197, 1899.

independent invariants. To $\sigma u$ belong $e_1$, $e_2$, $e_3$, and the periods $2\omega$, $2\omega'$. To $\bar{\sigma} v$ belong $\bar{e}_1$, $\bar{e}_2$, $\bar{e}_3$, and the periods $2\bar{\omega}$, $2\bar{\omega}'$.

We shall further use the notations

$$\begin{aligned} \omega &= \omega_1, & \bar{\omega} &= \bar{\omega}_1, \\ \omega + \omega' &= \omega_2, & \bar{\omega} + \bar{\omega}' &= \bar{\omega}_2, \\ \omega' &= \omega_3, & \bar{\omega}' &= \bar{\omega}_3, \end{aligned}$$

also

$$\begin{aligned} a^2 &= e_2 - e_3, & \bar{a}^2 &= \bar{e}_2 - \bar{e}_3, \\ b^2 &= e_3 - e_1, & \bar{b}^2 &= \bar{e}_3 - \bar{e}_1, \\ c^2 &= e_1 - e_2, & \bar{c}^2 &= \bar{e}_1 - \bar{e}_2. \end{aligned}$$

3. If we put $\dfrac{x_1}{x_0} = x, \quad \dfrac{x_2}{x_0} = y, \quad \dfrac{x_3}{x_0} = z,$
we get*

$$\begin{aligned} x^2 &= (\wp u - e_1)(\bar{\wp} v - \bar{e}_1), \\ y^2 &= (\wp u - e_2)(\bar{\wp} v - \bar{e}_2), \\ z^2 &= (\wp u - e_3)(\bar{\wp} v - \bar{e}_3), \end{aligned}$$

whence solving for $\wp u$, $\bar{\wp} v$ and $\wp u \bar{\wp} v$ we get

$$\begin{aligned} \wp u &= \phi(x, y, z), \\ \bar{\wp} v &= \psi(x, y, z), \\ \wp u \bar{\wp} v &= \chi(x, y, z), \end{aligned}$$

where $\phi$, $\psi$, and $\chi$ are functions of $x$, $y$, $z$ of the form

$$Ax^2 + By^2 + Cz^2 + D.$$

The equation of the surface is then

$$\chi = \phi\psi.$$

The surface is thus seen to be of the fourth degree. The Cartesian equation may be obtained in a simpler way by using the results of §6. If $u$ is fixed while $v$ varies the point $(x, y, z)$ lies on the intersection of two quadrics, and similarly when $v$ is fixed while $u$ varies. The "parametric lines" are therefore elliptic twisted quartics.

4. It is important to determine the region in which $u$ and $v$ are to vary in order to obtain the whole surface. In other words we must determine what

* See Schwarz, Formeln und Lehrsätze, Art. 18.

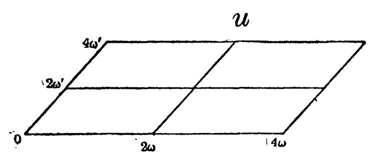
values of $u$, $v$ correspond to a given value of $x$, $y$, $z$. From the preceding paragraph

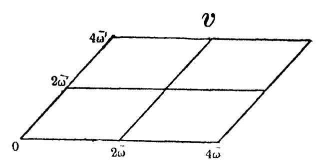
$$\wp u = \phi\,(x, y, z),$$
$$\bar{\wp} v = \psi\,(x, y, z),$$

so for given values $x_0$, $y_0$, $z_0$ of $x$, $y$, $z$ we get two sets of values;

$$u = \pm\, u_0 + 2\mu\omega + 2\mu'\omega',$$
$$v = \pm\, v_0 + 2\nu\bar{\omega} + 2\nu'\bar{\omega}',$$

where $u_0$, $v_0$ are any pair of values of $u$, $v$ which give $x_0$, $y_0$, $z_0$ while $\mu$, $\mu'$, $\nu$, $\nu'$ are integers. It is not necessary to consider values of $u$ and $v$ outside the four parallelograms as shown

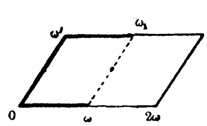


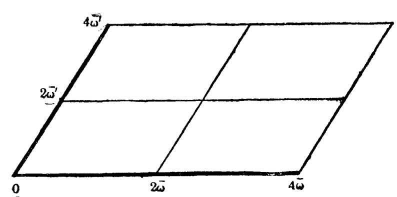




because $4\omega$, $4\omega'$ and $4\bar{\omega}$, $4\bar{\omega}'$ are periods for all the coordinates. Now $\phi$ and $\psi$ were functions involving only the squares of $x$, $y$, $z$, so all the values

$$\pm\, u_0 + 2\mu\omega + 2\mu'\omega',$$
$$\pm\, v_0 + 2\nu\bar{\omega} + 2\nu'\bar{\omega}',$$

of $u$ and $v$ will give correct values of $x_0$, $y_0$, $z_0$ to sign *près*. It is found that by applying formulæ 7 of article 18, Schwarz "Formeln," that if we take any one of the eight possible values of $u$ and combine it with all the eight possible values of $v$ lying in the four parallelograms above we get all the eight possibilities of sign. We may thus arbitrarily choose our $u$ in the lower half of the first parallelogram and allow $v$ all the four parallelograms. As to the points on the boundaries in the $v$ plane the usual conventions apply and we may omit the upper and right hand boundaries altogether. In the $u$ plane the only boundary necessary is from the origin up to and including the points $\omega$ and $\omega'$ and from $\omega'$ up to and including $\omega + \omega' = \omega_2$.



4. It is important to note the effect of certain transformations of the form

$$u' = u + \varepsilon\omega_\lambda,$$
$$v' = v + \varepsilon\bar{\omega}_\lambda + 2\varkappa\bar{\omega}_i,$$

$\varepsilon = 0$ or $1$, $\lambda = 1, 2, 3$, $i = 1, 2, 3$, $\varkappa =$ an integer. Restricting ourselves to the range of values indicated in the preceding paragraph we have the following sixteen transformations:

$$
\begin{array}{lll}
1^\circ. & u' = u, & v' = v, \\
2^\circ. & u' = u, & v' = v + 2\bar{\omega}_1, \\
3^\circ. & u' = u, & v' = v + 2\bar{\omega}_2, \\
4^\circ. & u' = u, & v' = v + 2\bar{\omega}_3, \\[6pt]
5^\circ. & u' = u + \omega_1, & v' = v + \bar{\omega}_1, \\
6^\circ. & u' = u + \omega_1, & v' = v + \bar{\omega}_1 + 2\bar{\omega}_1, \\
7^\circ. & u' = u + \omega_1, & v' = v + \bar{\omega}_1 + 2\bar{\omega}_2, \\
8^\circ. & u' = u + \omega_1, & v' = v + \bar{\omega}_1 + 2\bar{\omega}_3, \\[6pt]
9^\circ. & u' = u + \omega_2, & v' = v + \bar{\omega}_2, \\
10^\circ. & u' = u + \omega_2, & v' = v + \bar{\omega}_2 + 2\bar{\omega}_1, \\
11^\circ. & u' = u + \omega_2, & v' = v + \bar{\omega}_2 + 2\bar{\omega}_2, \\
12^\circ. & u' = u + \omega_2, & v' = v + \bar{\omega}_2 + 2\bar{\omega}_3, \\[6pt]
13^\circ. & u' = u + \omega_3, & v' = v + \bar{\omega}_3, \\
14^\circ. & u' = u + \omega_3, & v' = v + \bar{\omega}_3 + 2\bar{\omega}_1, \\
15^\circ. & u' = u + \omega_3, & v' = v + \bar{\omega}_3 + 2\bar{\omega}_2, \\
16^\circ. & u' = u + \omega_3, & v' = v + \bar{\omega}_3 + 2\bar{\omega}_3,
\end{array}
$$

The effect of these transformations on the coordinates is exhibited in the following table which is easily constructed by applying formulæ (2) of article 22, and formulæ 6 of article 18, Schwarz, "Formeln."

| | $x_0$ | $x_1$ | $x_2$ | $x_3$ |
|---|---|---|---|---|
| 1° | $x_0$ | $x_1$ | $x_2$ | $x_3$ |
| 2° | $-x_0$ | $-x_1$ | $x_2$ | $x_3$ |
| 3° | $-x_0$ | $x_1$ | $-x_2$ | $x_3$ |
| 4° | $-x_0$ | $x_1$ | $x_2$ | $-x_3$ |
| 5° | $x_1$ | $-b\bar{b}c\bar{c}\,x_0$ | $c\bar{c}\,x_3$ | $b\bar{b}\,x_2$ |
| 6° | $-x_1$ | $b\bar{b}c\bar{c}\,x_0$ | $c\bar{c}\,x_3$ | $b\bar{b}\,x_2$ |
| 7° | $-x_1$ | $-b\bar{b}c\bar{c}\,x_0$ | $-c\bar{c}\,x_3$ | $b\bar{b}\,x_2$ |
| 8° | $-x_1$ | $-b\bar{b}c\bar{c}\,x_0$ | $c\bar{c}\,x_3$ | $-b\bar{b}\,x_2$ |
| 9° | $x_2$ | $c\bar{c}\,x_3$ | $-c\bar{c}a\bar{a}\,x_0$ | $a\bar{a}\,x_1$ |
| 10° | $-x_2$ | $-c\bar{c}\,x_3$ | $-c\bar{c}a\bar{a}\,x_0$ | $a\bar{a}\,x_1$ |
| 11° | $-x_2$ | $c\bar{c}\,x_3$ | $c\bar{c}a\bar{a}\,x_0$ | $a\bar{a}\,x_1$ |
| 12° | $-x_2$ | $c\bar{c}\,x_3$ | $-c\bar{c}a\bar{a}\,x_0$ | $-a\bar{a}\,x_1$ |
| 13° | $x_3$ | $b\bar{b}\,x_2$ | $a\bar{a}\,x_1$ | $-a\bar{a}bb\,x_0$ |
| 14° | $-x_3$ | $-b\bar{b}\,x_2$ | $a\bar{a}\,x_1$ | $-a\bar{a}b\bar{b}\,x_0$ |
| 15° | $-x_3$ | $b\bar{b}\,x_2$ | $-a\bar{a}\,x_1$ | $-a\bar{a}b\bar{b}\,x_0$ |
| 16° | $-x_3$ | $b\bar{b}\,x_2$ | $a\bar{a}\,x_1$ | $a\bar{a}b\bar{b}\,x_0$ |

5. It is not difficult to see that the above sixteen transformations form a group, and it follows that when we have found a relation between any or all of the coordinates we can at once derive sixteen other relations by operating on these coordinates with these transformations. The relations thus found need

not necessarily be distinct. Thus the equation of the surface will, of course, be invariant under all these transformations.

6. THEOREM. *The intersection of the surface with each of the planes of reference is a pair of conics. The three vertices of the tetrahedron of reference in any reference plane furnishes a triangle which is self-polar to both the conics in that plane.*

To prove this, put $x_0 = 0$. This gives $u = 0$ or $v = 0$. Taking $u = 0$, we have

$$\begin{aligned} x_0 &= 0\ , \\ x_1 &= \bar{\sigma}_1 v, \\ x_2 &= \bar{\sigma}_2 v, \\ x_3 &= \bar{\sigma}_3 v, \end{aligned}$$

whence

$$a^2x_1^2 + b^2x_2^2 + c^2x_3^2 = a^2\bar{\sigma}_1^2 v + b^2\bar{\sigma}_2^2 v + c^2\sigma_3^2 v = 0$$

(Schwarz, "Formeln," article 24.) Our conic is thus

$$a^2x_1^2 + b^2x_2^2 + c^2x_3^2 = 0.$$

The other conic in this plane is obtained by putting $v = 0$ and turns out to be

$$\bar{a}^2x_1^2 + \bar{b}^2x_2^2 + \bar{c}^2x_3^2 = 0.$$

The sections by the three other coordinate planes are obtained by operating on the above with the transformations of our group. The sections by the four planes are

$$\begin{aligned} x_0 &= 0, \quad (a^2x_1^2 + b^2x_2^2 + c^2x_3^2)(\bar{a}^2x_1^2 + \bar{b}^2x_2^2 + \bar{c}^2x_3^2) &&= 0, \\ x_1 &= 0, \quad (a^2\bar{b}^2\bar{c}^2x_0^2 + \bar{b}^2x_2^2 + \bar{c}^2x_3^2)(\bar{a}^2b^2c^2x_0^2 + b^2x_2^2 + c^2x_3^2) &&= 0, \\ x_2 &= 0, \quad (\bar{a}^2b^2\bar{c}^2x_0^2 + \bar{a}^2x_1^2 + \bar{c}^2x_3^2)(a^2\bar{b}^2c^2x_0^2 + a^2x_1^2 + c^2x_3^2) &&= 0, \\ x_3 &= 0, \quad (\bar{a}^2\bar{b}^2c^2x_0^2 + \bar{a}^2x_1^2 + \bar{b}^2x_2^2)(a^2b^2\bar{c}^2x_0^2 + a^2x_1^2 + b^2x_2^2) &&= 0, \end{aligned}$$

These conics are seen to be referred to their self-polar triangles.

7. By means of the intersections of the surface with the planes of reference given above we may easily derive the equation of the surface in tetrahedral coordinates.

From paragraph 3 we note that the equation is of the form

$$\sum_{i,j=0,1,2,3} A_{ij}\, x_i^2\, x_j^2 = 0.$$

Put successively $x_0 = 0$, $x_1 = 0$, $x_2 = 0$, $x_3 = 0$ in this equation and identify with the above intersections. The equation of the surface thus obtained is

$$\begin{aligned} a^2\bar{a}^2 b^2\bar{b}^2 c^2\bar{c}^2 x_0^4 &+ a^2\bar{a}^2 x_1^4 + b^2\bar{b}^2 x_2^4 + c^2\bar{c}^2 x_3^4 \\ &+ a^2\bar{a}^2 (c^2\bar{b}^2 + \bar{c}^2 b^2)\, x_0^2 x_1^2 \\ &+ b^2\bar{b}^2 (a^2\bar{c}^2 + \bar{a}^2 c^2)\, x_0^2 x_2^2 \\ &+ c^2\bar{c}^2 (b^2\bar{a}^2 + \bar{b}^2 a^2)\, x_0^2 x_3^2 \\ &+ \qquad (a^2\bar{b}^2 + \bar{a}^2 b^2)\, x_1^2 x_2^2 \\ &+ \qquad (b^2\bar{c}^2 + \bar{b}^2 c^2)\, x_2^2 x_3^2 \\ &+ \qquad (c^2\bar{a}^2 + \bar{c}^2 a^2)\, x_3^2 x_1^2 = 0. \end{aligned}$$

Study has expressed this in a somewhat more elegant form by writing

$$\begin{aligned} x_0' &= -\sqrt{a\bar{a}\, b\bar{b}\, c\bar{c}}\; x_0, \\ x_1' &= \sqrt{a\bar{a}}\; x_1, \\ x_2 &= \sqrt{b\bar{b}}\; x_2, \\ x_3 &= \sqrt{c\bar{c}}\; x_3, \\ a/a &= a, \\ b/\bar{b} &= b, \\ c/\bar{c} &= c. \end{aligned}$$

We thus get, dropping accents,

$$\begin{aligned} x_0^4 + x_1^4 + x_2^4 + x_3^4 &+ \left(\frac{c}{b} + \frac{b}{c}\right)(x_0^2 x_1^2 + x_2^2 x_3^2) \\ &+ \left(\frac{a}{c} + \frac{c}{a}\right)(x_0^2 x_2^2 + x_3^2 x_1^2) \\ &+ \left(\frac{a}{b} + \frac{b}{a}\right)(x_0^2 x_3^2 + x_1^2 x_2^2) = 0. \end{aligned}$$

From this, the equation of Fresnel's "Wave Surface" may be easily derived by the transformation

$$\begin{aligned} x &= \sqrt{-ab}\; \frac{x_1}{x_0}, \\ y &= \sqrt{-bc}\; \frac{x_2}{x_0}, \\ z &= \sqrt{-ca}\; \frac{x_3}{x_0}. \end{aligned}$$

8. *Singular Planes.* Starting from the identities (Schwarz, "Formeln," article 24),

$$a^2\sigma_1^2u + b^2\sigma_2^2u = -c^2\sigma_3^2u,$$
$$\bar{a}^2\bar{\sigma}_1^2v + \bar{b}^2\bar{\sigma}_2^2v = -\bar{c}^2\bar{\sigma}_3^2v,$$

and using the well-known formula

$$(P^2+Q^2)(R^2+S^2) = (PR+QS)^2 + (PS-QR)^2,$$

we get

$$(a\bar{a}\sigma_1u\bar{\sigma}_1v + b\bar{b}\sigma_2u\bar{\sigma}_2v)^2 + (a\bar{b}\sigma_1u\bar{\sigma}_2v - \bar{a}b\bar{\sigma}_1v\sigma_2u)^2 = c^2\bar{c}^2\sigma_3^2u\bar{\sigma}_3^2v$$

or

$$c^2\bar{c}^2\sigma_3^2u\bar{\sigma}_3^2v - (a\bar{a}\sigma_1u\bar{\sigma}_1v + b\bar{b}\sigma_2u\bar{\sigma}_2v)^2 = (a\bar{b}\sigma_1u\bar{\sigma}_2v - \bar{a}b\bar{\sigma}_1v\sigma_2u)^2,$$

whence,

$$(c\bar{c}\sigma_3u\bar{\sigma}_3v - a\bar{a}\sigma_1u\bar{\sigma}_1v - b\bar{b}\sigma_2u\bar{\sigma}_2v)(c\bar{c}\sigma_3u\bar{\sigma}_3v + a\bar{a}\sigma_1u\bar{\sigma}_1v + b\bar{b}\sigma_2u\bar{\sigma}_2v)$$
$$= (a\bar{b}\sigma_1u\bar{\sigma}_2v - \bar{a}b\bar{\sigma}_1v\sigma_2u)^2.$$

Now, the identity which we started with holds also when $\sigma_2u$ is changed to $-\sigma_2u$. We thus get the identity

$$(c\bar{c}\sigma_3u\bar{\sigma}_3v - a\bar{a}\sigma_1u\bar{\sigma}_1v + b\bar{b}\sigma_2u\bar{\sigma}_2v)(c\bar{c}\sigma_3u\bar{\sigma}_3v + a\bar{a}\sigma_1u\bar{\sigma}_1v - b\bar{b}\sigma_2u\bar{\sigma}_2v)$$
$$= (a\bar{b}\sigma_1u\bar{\sigma}_2v + \bar{a}b\bar{\sigma}_1v\sigma_2u).$$

Multiplying these last two equations together, member by member, we have

$$(c\bar{c}x_3 - a\bar{a}x_1 - b\bar{b}x_2)(c\bar{c}x_3 + a\bar{a}x_1 + b\bar{b}x_2)(c\bar{c}x_3 - a\bar{a}x_1 + b\bar{b}x_2)(c\bar{c}x_3 + a\bar{a}x_1 - b\bar{b}x_2)$$
$$= (a^2\bar{b}^2\sigma_1^2u\bar{\sigma}_2^2v - \bar{a}^2b^2\bar{\sigma}_1^2v\sigma_2^2u)^2.$$

We may identify this equation of the tetrahedroid with that given in paragraph 7. Putting the right side of this last equation in the form

$$(A_0x_0^2 + A_1x_1^2 + A_2x_2^2 + A_3x_3^2)^2$$

we get, comparing coefficients,

$$\frac{A_0^2}{a^2\bar{a}^2b^2\bar{b}^2c^2\bar{c}^2} = \frac{A_1^2 - a^4\bar{a}^4}{a^2\bar{a}^2} = \frac{A_2^2 - b^4\bar{b}^4}{b^2\bar{b}^2} = \text{etc.} = \rho.$$

By using the relations $a^2 + b^2 + c^2 = 0$, $\bar{a}^2 + \bar{b}^2 + \bar{c}^2 = 0$, it is not difficult to obtain the value of $\rho$ in the simple form

$$\rho = \frac{4a^2\bar{a}^2b^2\bar{b}^2c^2\bar{c}^2}{(\bar{b}^2c^2 - b^2\bar{c}^2)^2},$$

whence,

$$A_0 = \frac{2a^2\bar{a}^2b^2\bar{b}^2c^2\bar{c}^2}{\bar{b}^2c^2 - b^2\bar{c}^2},$$

$$A_1 = a^2\bar{a}^2 \frac{\bar{b}^2c^2 + b^2\bar{c}^2}{\bar{b}^2c^2 - b^2\bar{c}^2},$$

$$A^2 = b^2\bar{b}^2 \frac{\bar{c}^2a^2 + c^2\bar{a}^2}{\bar{b}^2c^2 - b^2\bar{c}^2},$$

$$A_3 = c^2\bar{c}^2 \frac{\bar{a}^2b^2 + a^2\bar{b}^2}{\bar{b}^2c^2 - b^2\bar{c}^2}.$$

We have thus thrown the equation of the surface in the form

$$q_1q_2q_3q_4 = Q_0^2,$$

where

$$\begin{aligned} q_1 &= \phantom{-}a\bar{a}x_1 + b\bar{b}x_2 + c\bar{c}x_3, \\ q_2 &= -a\bar{a}x_1 + b\bar{b}x_2 + c\bar{c}x_3, \\ q_3 &= -a\bar{a}x_1 - b\bar{b}x_2 + c\bar{c}x_3, \\ q_4 &= \phantom{-}a\bar{a}x_1 - b\bar{b}x_2 + c\bar{c}x_3, \end{aligned}$$

and

$$Q_0 = \frac{1}{\bar{b}^2c^2 - b^2\bar{c}^2}\Big[2a^2\bar{a}^2b^2\bar{b}^2c^2\bar{c}^2x_0^2 + a^2\bar{a}^2(\bar{b}^2c^2 + b^2\bar{c}^2)x_1^2 + b^2\bar{b}^2(\bar{c}^2a^2 + c^2\bar{a}^2)x_2^2 + c^2\bar{c}^2(\bar{a}^2b^2 + a^2\bar{b}^2)x_3^2\Big]$$

From the general theory of surfaces, each of the planes $q_i = 0$ is tangent to the tetrahedroid along its intersection with the quadric surface $Q_0 = 0$. We have, in fact, found four singular tangent planes which touch the tetrahedroid along a conic counted twice.

Apply, now, the operations of the group. We thus derive sixteen singular tangent planes, all of which touch the surface along a conic counted twice. Writing, in general, $q(a, b, c, d)$ for the plane $ax_0 + bx_1 + cx_2 + dx_3 = 0$, our singular tangent planes are

$$\begin{array}{llll} q(\ 0, & a\bar{a}, & \pm b\bar{b}, & \pm c\bar{c}), \\ q(a\bar{a}, & 0, & \pm 1, & \pm 1), \\ q(b\bar{b}, & \pm 1, & 0, & \pm 1), \\ q(c\bar{c}, & \pm 1, & \pm 1, & 0). \end{array}$$

The four quadric surfaces $Q_0$, $Q_1$, $Q_2$, $Q_3$ are seen to be referred to their self-polar tetrahedrons. They are:

$$\begin{aligned}
Q_0 \equiv 2a^2\bar{a}^2b^2\bar{b}^2c^2\bar{c}^2x_0^2 &+ a^2\bar{a}^2(\bar{b}^2c^2 + b^2\bar{c}^2)\,x_1^2 \\
&+ b^2\bar{b}^2(\bar{c}^2a^2 + c^2\bar{a}^2)\,x_2^2 \\
&+ c^2\bar{c}^2(\bar{a}^2b^2 + a^2\bar{b}^2)\,x_3^2 = 0,
\end{aligned}$$

$$\begin{aligned}
Q_1 \equiv 2a^2\bar{a}^2x_1^2 &+ a^2\bar{a}^2(\bar{b}^2c^2 + b^2\bar{c}^2)\,x_0^2 \\
&+ (\bar{c}^2a^2 + c^2\bar{a}^2)\,x_3^2 \\
&+ (\bar{a}^2b^2 + a^2\bar{b}^2)\,x_2^2 = 0,
\end{aligned}$$

$$\begin{aligned}
Q_2 \equiv 2b^2\bar{b}^2x_2^2 &+ (\bar{b}^2c^2 + b^2\bar{c}^2)\,x_3^2 \\
&+ b^2\bar{b}^2(\bar{c}^2a^2 + c^2\bar{a}^2)\,x_0^2 \\
&+ (\bar{a}^2b^2 + a^2\bar{b}^2)\,x_1^2 = 0,
\end{aligned}$$

$$\begin{aligned}
Q_3 \equiv 2c^2\bar{c}^2x_3^2 &+ (\bar{b}^2c^2 + b^2\bar{c}^2)\,x_2^2 \\
&+ (\bar{c}^2a^2 + c^2\bar{a}^2)\,x_1^2 \\
&+ c^2\bar{c}^2(\bar{a}^2b^2 + a^2\bar{b}^2)\,x_0^2 = 0.
\end{aligned}$$

The above method of obtaining the singular planes of the surface is used by Lacour in his treatment of Fresnel's "Wave Surface" (Nouvelles Annales, third series, Vol. XVII, p. 266).

A glance at the equation of the tetrahedroid shows that

$$\begin{aligned}
x_0\,Q_0 &= \frac{\partial f}{\partial x_0}, \\
x_1\,Q_1 &= \frac{\partial f}{\partial x_1}, \\
x_2\,Q_2 &= \frac{\partial f}{\partial x_2}, \\
x_3\,Q_3 &= \frac{\partial f}{\partial x_3}.
\end{aligned}$$

The tangent plane to the tetrahedroid is therefore given parametrically as follows:

$$\begin{aligned}
\rho u_0 &= x_0'\,Q_0', \\
\rho u_1 &= x_1'\,Q_1', \\
\rho u_2 &= x_2'\,Q_2', \\
\rho u_3 &= x_3'\,Q_3',
\end{aligned}$$

or in terms of $u$ and $v$,

$$
\begin{aligned}
\rho u_0 &= \sigma u \bar{\sigma} v \,(a^2 \bar{b}^2 \sigma_1^2 u \bar{\sigma}_2^2 v - \bar{a}^2 b^2 \bar{\sigma}_1^2 v \sigma_2^2 u), \\
\rho u_1 &= \sigma_1 u \bar{\sigma}_1 v \,(\bar{a}^2 \sigma_2^2 u \bar{\sigma}^2 v - a^2 \sigma^2 u \bar{\sigma}_2^2 v), \\
\rho u_2 &= \sigma_2 u \bar{\sigma}_2 v \,(\bar{b}^2 \sigma_3^2 u \bar{\sigma}^2 v - b^2 \sigma^2 u \bar{\sigma}_3^2 v), \\
\rho u_3 &= \sigma_3 u \bar{\sigma}_3 v \,(\bar{c}^2 \sigma_1^2 u \bar{\sigma}^2 v - c^2 \sigma^2 u \bar{\sigma}_1^2 v).
\end{aligned}
$$

By using the formulæ of art. 24, Schwarz, " Formeln," these equations may be written

$$
\begin{aligned}
\rho u_0 &= \sigma u \bar{\sigma} v \,(a^2 \bar{b}^2 \sigma_1^2 u \bar{\sigma}_2^2 v - \bar{a}^2 b^2 \sigma_2^2 u \bar{\sigma}_1^2 v), \\
\rho u_1 &= \sigma_1 u \bar{\sigma}_1 v \,(\sigma_2^2 u \bar{\sigma}_3^2 v - \sigma_3^2 u \bar{\sigma}_2^2 v), \\
\rho u_2 &= \sigma_2 u \bar{\sigma}_2 v \,(\sigma_3^2 u \bar{\sigma}_1^2 v - \sigma_1^2 u \bar{\sigma}_3^2 v), \\
\rho u_3 &= \sigma_3 u \bar{\sigma}_3 v \,(\sigma_1^2 u \bar{\sigma}_2^2 v - \sigma_2^2 u \bar{\sigma}_1^2 v).
\end{aligned}
$$

These equations give us the parametric representation of the reciprocal surface. Treating the $u$'s as point-coordinates, we find the trace on the plane $u_0 = 0$ is the pair of conics

$$
\left(\frac{u_1^2}{a^2} + \frac{u_2^2}{b^2} + \frac{u_3^2}{c_2}\right)\left(\frac{u_1^2}{\bar{a}^2} + \frac{u_2^2}{\bar{b}^2} + \frac{u_3^2}{\bar{c}^2}\right) = 0,
$$

with similar results for the other coordinate planes. The equation of the reciprocal surface is found to be

$$
\begin{aligned}
&\frac{u_0^4}{a^2\bar{a}^2 b^2\bar{b}^2 c^2\bar{c}^2} + \frac{u_1^4}{a^2\bar{a}^2} + \frac{u_2^4}{b^2\bar{b}^2} + \frac{u_3^4}{c^2\bar{c}^2} + \left(\frac{1}{b^2\bar{c}^2} + \frac{1}{\bar{b}^2 c^2}\right)\frac{u_0^2 u_1^2}{a^2\bar{a}^2} \\
&\quad + \left(\frac{1}{c^2\bar{a}^2} + \frac{1}{\bar{c}^2 a^2}\right)\frac{u_0^2 u_2^2}{b^2\bar{b}^2} + \left(\frac{1}{a^2\bar{b}^2} + \frac{1}{\bar{a}^2 b^2}\right)\frac{u_0^2 u_3^2}{c^2\bar{c}^2} + \left(\frac{1}{b^2\bar{c}^2} + \frac{1}{\bar{b}^2 c^2}\right) u_2^2 u_3^2 \\
&\quad + \left(\frac{1}{c^2\bar{a}^2} + \frac{1}{\bar{c}^2 a^2}\right) u_3^2 u_1^2 + \left(\frac{1}{a^2\bar{b}^2} + \frac{1}{\bar{a}^2 b^2}\right) u_1^2 u_2^2 = 0.
\end{aligned}
$$

This is again a tetrahedroid surface. It may be obtained from the first by the transformation

$$
\begin{aligned}
u_0 &= a\bar{a}\, b\bar{b}\, c\bar{c}\, x_0, \\
u_1 &= a\bar{a}\, x_1, \\
u_2 &= b\bar{b}\, x_2, \\
u_3 &= c\bar{c}\, x_3.
\end{aligned}
$$

9. *Singular Points.* The singular points of the tetrahedroid may be found in the usual manner by obtaining the values of $u$ and $v$, for which we have simultaneously $u_i = 0$ $(i = 0, 1, 2, 3)$. They may be more readily found, however, by making use of the fact that singular planes reciprocate into singular points,

and vice versa. Reciprocating our sixteen singular tangent planes, we obtain at once the sixteen singular points

$$\begin{array}{llll} 0, & 1, & \pm 1, & \pm 1, \\ 1, & 0, & \pm c\bar{c}, & \pm b\bar{b}, \\ 1, & \pm c\bar{c}, & 0, & \pm a\bar{a}, \\ 1, & \pm b\bar{b}, & \pm a\bar{a}, & 0. \end{array}$$

10. It is seen that there are four singular points in each plane of reference just as there are four singular planes through each vertex of the tetrahedron of reference.

It is easily verified, moreover, that the four singular points in any plane of reference are the points where the two conics in that plane intersect. Also the line joining any two of these points in a plane of reference passes through a vertex of the tetrahedron of reference. By reciprocating, we find that the four singular planes, through any vertex of the tetrahedron of reference, intersect along lines which lie two in each of the planes of reference through that vertex. These four planes also cut the opposite plane of reference in the four common tangents of the two conics in that plane. Their points of tangency lie by twos on straight lines which also pass through a vertex of the tetrahedron of reference. These theorems, of course, are easily established directly by using the equations of the planes.

11. The singular point (0, 1, 1, 1) lies in each of the six singular planes,

$$\begin{array}{llll} q\,(a\bar{a}, & 0, & +1, & -1), \\ q\,(a\bar{a}, & 0, & -1, & +1), \\ q\,(b\bar{b}, & +1, & 0, & -1), \\ q\,(b\bar{b}, & -1, & 0, & +1), \\ q\,(c\bar{c}, & +1, & -1, & 0), \\ q\,(c\bar{c}, & -1, & +1, & 0) \end{array}$$

Making use of the transformations of the group of paragraph 3, we have the theorem :

*Through every singular point pass six singular tangent planes.*

Also, by reciprocation, or directly from the coordinates of the singular points;

*There are six singular points in every singular tangent plane.*

A convenient notation for the sixteen singular planes is the following:

$$\begin{aligned}
Ia &= q\,(0, & +a\bar{a}, & +b\bar{b}, & +c\bar{c}),\\
Ib &= q\,(0, & +a\bar{a}, & +b\bar{b}, & -c\bar{c}),\\
Ic &= q\,(0, & +a\bar{a}, & -b\bar{b}, & +c\bar{c}),\\
Id &= q\,(0, & +a\bar{a}, & -b\bar{b}, & -c\bar{c}),
\end{aligned}$$

$$\begin{aligned}
IIa &= q\,(a\bar{a}, & 0, & +1, & +1);\\
IIb &= q\,(a\bar{a}, & 0, & +1, & -1);\\
IIc &= q\,(a\bar{a}, & 0, & -1, & +1);\\
IId &= q\,(a\bar{a}, & 0, & -1, & -1);
\end{aligned}$$

$$\begin{aligned}
IIIa &= q\,(b\bar{b}, & +1, & 0, & +1);\\
IIIb &= q\,(b\bar{b}, & +1, & 0, & -1);\\
IIIc &= q\,(b\bar{b}, & -1, & 0, & +1);\\
IIId &= q\,(b\bar{b}, & -1, & 0, & -1);
\end{aligned}$$

$$\begin{aligned}
IVa &= q\,(c\bar{c}, & +1, & +1, & 0);\\
IVb &= q\,(c\bar{c}, & +1, & -1, & 0);\\
IVc &= q\,(c\bar{c}, & -1, & +1, & 0);\\
IVd &= q\,(c\bar{c}, & -1, & -1, & 0);
\end{aligned}$$

and similarly for the singular points:

$$\begin{aligned}
1a &= (0, & 1, & +1, & +1);\\
1b &= (0, & 1, & +1, & -1);
\end{aligned}$$

etc.

$$2a = (1,\ 0,\ +c\bar{c}\ \ +b\bar{b});$$

etc.

$$3a = (1,\ \ +c\bar{c}\ \ 0,\ \ a\bar{a});$$

etc.

$$4a = (1,\ \ b\bar{b}\ \ \ a\bar{a}\ \ 0);$$

etc.

With this notation we may write down the six singular points in each singular plane as follows:

$$\begin{aligned}
Ia&: \ 2b,\ 2c,\ 3b,\ 3c,\ 4b,\ 4c;\\
Ib&: \ 2a,\ 2d,\ 3a,\ 3d,\ 4b,\ 4c;\\
Ic&: \ 2a,\ 2d,\ 3b,\ 3c,\ 4a,\ 4d;\\
Id&: \ 2b,\ 2c,\ 3a,\ 3d,\ 4a,\ 4d;
\end{aligned}$$

| | | | | | | |
|---|---|---|---|---|---|---|
| $IIa$: | $3b$, | $3d$, | $4b$, | $4d$, | $1b$, | $1c$; |
| $IIb$: | $3a$, | $3c$, | $4b$, | $4d$, | $1a$, | $1d$; |
| $IIc$: | $3b$, | $3d$, | $4a$, | $4c$, | $1a$, | $1d$; |
| $IId$: | $3a$, | $3c$, | $4a$, | $4c$, | $1b$, | $1c$; |
| $IIIa$: | $4c$, | $4d$, | $1b$, | $1d$, | $2b$, | $2d$; |
| $IIIb$: | $4c$, | $4d$, | $1a$, | $1c$, | $2a$, | $2c$; |
| $IIIc$: | $4a$, | $4b$, | $1a$, | $1c$, | $2b$, | $2d$; |
| $IIId$: | $4a$, | $4b$, | $1b$, | $1d$, | $2a$, | $2c$; |
| $IVa$: | $1c$, | $1d$, | $2c$, | $2d$, | $3c$, | $3d$; |
| $IVb$: | $1a$, | $1b$, | $2a$, | $2b$, | $3c$, | $3d$; |
| $IVc$: | $1a$, | $1b$, | $2c$, | $2d$, | $3a$, | $3b$; |
| $IVd$: | $1c$, | $1d$, | $2a$, | $2b$, | $3a$, | $3b$. |

The six singular planes through each singular point are:

| | | | | | | |
|---|---|---|---|---|---|---|
| $1a$: | $IIb$, | $IIc$, | $IIIb$, | $IIIc$, | $IVb$, | $IVc$; |
| $1b$: | $IIa$, | $IId$, | $IIIa$, | $IIId$, | $IVb$, | $IVc$; |
| $1c$: | $IIa$, | $IId$, | $IIIb$, | $IIIc$, | $IVa$, | $IVd$; |
| $1d$: | $IIb$, | $IIc$, | $IIIa$, | $IIId$, | $IVa$, | $IVd$; |
| $2a$: | $IIIb$, | $IIId$, | $IVb$, | $IVd$, | $Ib$, | $Ic$; |
| $2b$: | $IIIa$, | $IIIc$, | $IVb$, | $IVd$, | $Ia$, | $Id$; |
| $2c$: | $IIIb$, | $IIId$, | $IVa$, | $IVc$, | $Ia$, | $Id$; |
| $2d$: | $IIIa$, | $IIIc$, | $IVa$, | $IVc$, | $Ib$, | $Ic$; |
| $3a$: | $IVc$, | $IVd$, | $Ib$, | $Id$, | $IIb$, | $IId$; |
| $3b$: | $IVc$, | $IVd$, | $Ia$, | $Ic$, | $IIa$, | $IIc$; |
| $3c$: | $IVa$, | $IVb$, | $Ia$, | $Ic$, | $IIb$, | $IId$; |
| $3d$: | $IVa$, | $IVb$, | $Ib$, | $Id$, | $IIa$, | $IIc$; |
| $4a$: | $Ic$, | $Id$, | $IIc$, | $IId$, | $IIIc$, | $IIId$; |
| $4b$: | $Ia$, | $Ib$, | $IIa$, | $IIb$, | $IIIc$, | $IIId$; |
| $4c$: | $Ia$, | $Ib$, | $IIc$, | $IId$, | $IIIa$, | $IIIb$; |
| $4d$: | $Ic$, | $Id$, | $IIa$, | $IIb$, | $IIIa$, | $IIIb$. |

**12.** A number of theorems may be obtained by inspection from these two tables. In the first place, no three singular points lie in more than one singular plane, and no three singular planes pass through more than one singular point. This means that no three singular points are collinear and no three singular planes pass through one line. Again, there are two points common to any pair of planes, and two planes common to any pair of points. This means that the

120 lines joining the sixteen points are precisely the 120 lines in which the sixteen planes intersect.

The four planes $Ia\,Ib\,Ic\,Id$ all pass through the vertex $v_0=(1, 0, 0, 0)$ of the tetrahedron of reference. Therefore their six lines of intersection do also. Now, by looking at the table, we see that the points

$2b$ and $2c$ lie in the intersection of $Ia$ and $Id$;
$3b$ and $3c$ lie in the intersection of $Ia$ and $Ic$;
$4b$ and $4c$ lie in the intersection of $Ia$ and $Ib$.

These six points, therefore, lie by twos on three straight lines through the vertex $v_0$. Now, these six points all lie on the conic in $Ia$, and we have the theorem:

*In each singular plane the six singular points are three pairs in involution, and the center of involution is a vertex of the tetrahedron of reference.*

We have shown the above theorem for only one singular plane. It follows for the others by using the transformations of the group of paragraph 3.

To write the reciprocal theorem, it is necessary to see that the six planes through any singular point are tangent to a quadric cone. Consider, in fact, the six planes through $1a$. They are by the table $IIb$, $IIc$, $IIIb$, $IIIc$, $IVb$, $IVc$, or, written at length,

$$\begin{array}{rrrr}
q\,(a\bar{a}, & 0, & 1, & -1), \\
q\,(a\bar{a}, & 0, & -1, & 1), \\
q\,(b\bar{b}, & 1, & 0, & -1), \\
q\,(b\bar{b}, & -1, & 0, & 1), \\
q\,(c\bar{c}, & 1, & -1, & 0), \\
q\,(c\bar{c}, & -1, & 1, & 0).
\end{array}$$

The traces of these planes on the plane $x_1=0$ will give the six lines

$$\begin{array}{l}
a\bar{a}\,x_0 \pm x_2 \mp x_3 = 0 \\
b\bar{b}\,x_0 \qquad \mp x_3 = 0, \\
c\bar{c}\,x_0 \pm x_2 \qquad = 0.
\end{array}$$

It will suffice if we show that these lines are all tangent to the same conic. This requires the vanishing of the determinant

$$\begin{vmatrix}
a^2\bar{a}^2, & 1, & 1, & a\bar{a}, & -a\bar{a}, & -1 \\
a^2\bar{a}^2, & 1, & 1, & -a\bar{a}, & a\bar{a}, & -1 \\
b^2\bar{b}^2, & 0, & 1, & 0, & -b\bar{b}, & 0 \\
b^2\bar{b}^2, & 0, & 1, & 0, & b\bar{b}, & 0 \\
c^2\bar{c}^2, & 1, & 0, & -c\bar{c}, & 0, & 0 \\
c^2\bar{c}^2, & 1, & 0, & c\bar{c}, & 0, & 0
\end{vmatrix} = 0.$$

Subtract the first row from the second, the third from the fourth, the fifth from the sixth and the determinant is seen at once to be equal to zero.

The reciprocal theorem of the one last written reads therefore:

*The six singular planes through any singular point are tangent to a quadric cone and are three pairs in involution. The intersecting pairs meet in a plane of reference.*

13. Consider any pair of singular points not in the same face of the tetrahedron of reference, e. g. $1c$, $4d$, we see by the table that these lie in the intersection of the two singular planes $IIa$ and $IIIb$. The other four points in $IIa$ are $3b$, $3d$, $4b$, and $1b$. The other four in $IIIb$ are $4c$, $1a$ $2a$, and $2c$. Now, the line joining $3b$ and $3d$ meets the line joining $1a$ and $4c$, since these two lines are in $IIc$. In general,

$$
\begin{aligned}
&\overline{1b,\ 3b} \text{ meets } \overline{1a,\ 2c} \text{ in } IVc,\\
&\overline{1b,\ 3d} \text{ meets } \overline{1a,\ 2a} \text{ in } IVb,\\
&\overline{1b,\ 4b} \text{ meets } \overline{2a,\ 2c} \text{ in } IIId,\\
&\overline{3b,\ 3d} \text{ meets } \overline{1a,\ 4c} \text{ in } IIc,\\
&\overline{3b,\ 4b} \text{ meets } \overline{2c,\ 4c} \text{ in } Ia,\\
&\overline{3d,\ 4b} \text{ meets } \overline{2a,\ 4c} \text{ in } Ib.
\end{aligned}
$$

Naturally these lines intersect on the line $\overline{1c,\ 4d}$. Thus the six lines of the complete quadrilateral in $IIa$ meet the line $\overline{1c,\ 4d}$ in the same points in which that line is met by the six lines of the complete quadrilateral in $IIIb$. A similar state of affairs exists when the two points selected lie both in the same face of the tetrahedron of reference except that here one vertex of the complete quadrilateral in each of the two singular planes will lie in the line joining the two chosen singular points, being in fact the vertex of the tetrahedron of reference lying in that line. From these considerations we have the theorem:

*Every line joining two singular points meets twelve other such lines in six points which are in involution. If the two singular points lie in the same face of the tetrahedron of reference, two of these six points coincide with a vertex of the tetrahedron of reference, which is a double point of the involution.*

This theorem is its own reciprocal.

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